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Common Misperceptions of the Hyperbolic Heat Equation

T. J. Bright* and Z. M. Zhang†
Georgia Institute of Technology, Atlanta, Georgia 30332

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The Cattaneo equation has been proposed as a more general form of Fourier's law and, to date, many believe that Cattaneo's equation extends the validation regime of Fourier's law to time scales shorter than the relaxation time of a material. Cattaneo's equation leads to a form of the heat equation known as the hyperbolic heat equation, which is a damped-wave equation and predicts that heat will propagate in waves with a finite speed. Because of lack of experimental evidence and no sound derivation of hyperbolic heat equation, there is simply no justification for accepting Cattaneo's equation. Experiments claiming results corresponding to hyperbolic heat equation near room temperature in nonhomogenous media are misleading and more than likely due to improper specification of the problem or other phenomena associated with the porous materials used in the experiment. Accepting hyperbolic heat equation simply because it predicts a finite speed of propagation is fundamentally misleading with regard to conduction heat transfer at short time scales. This paper provides a point-to-point clarification, including statistical theories, equilibrium and irreversible thermodynamics, and the experimental aspect, as regards to the common misunderstandings of hyperbolic heat equation by many past and contemporary heat transfer researchers.

Nomenclature

$\mathbf{a}$	=	acceleration vector, $\mathrm{m/s^2}$
c_p	=	specific heat capacity at constant pressure, $\mathrm{J/kg \cdot K}$
f	=	distribution function
f_0	=	equilibrium distribution function
k	=	thermal conductivity
$\mathbf{q}''$	=	heat flux vector, $\mathrm{W/m^2}$
$\dot{q}$	=	volumetric heat generation, $\mathrm{W/m^3}$
$\mathbf{r}$	=	position vector, m
S	=	nondimensional heat source term
T	=	temperature, K
t	=	time, s
$\mathbf{v}$	=	velocity vector, m/s
v	=	magnitude of velocity or speed, m/s
x	=	position, m
α	=	thermal diffusivity, $k/\rho c_p$
ζ	=	similarity parameter, $x/\sqrt{4\alpha t}$
η	=	dimensionless position
θ	=	dimensionless temperature
ξ	=	dimensionless time
ρ	=	density, $\mathrm{kg/m^3}$
τ	=	relaxation time, s

Subscripts

s	=	surface
tw	=	temperature wave (commonly known as thermal wave)

I. Introduction

FOURIER'S law has long been established as the governing law of heat conduction, describing the thermal energy propagation in a medium via a diffusion process. It has served as a reliable model for predicting the temperature in a medium, as well as the rate of heat propagation through a medium, that has been validated by numerous experiments. One of the predicted results of Fourier's law, as with all diffusion processes, is that the effect of a source will be instantaneously felt everywhere in a medium, although such a nonzero effect is practically negligible at large distances. Since heat is carried by particles such as electrons, and quanta such as phonons, which are forbidden to propagate at speeds greater than that of light, it is impossible that the response to a sudden heat flux at one location in a medium should be instantaneously felt at all other locations within the medium. This paradox has spurred much academic interest in the last half century towards seeking a model that can predict a finite speed of propagation [1–5].

Fourier's law can be combined with the energy equation to give the conventional heat conduction equation or parabolic heat equation (PHE). PHE is a macroscopic description of the microscopic phenomena associated with heat diffusion and is an excellent approximation at length scales much greater than the mean free path and at time scales much greater than the thermal relaxation time, so that the local equilibrium assumption is applicable. Hyperbolic heat equation (HHE) is an attempt to extend the macroscopic description of heat transfer to very short time scales at which Fourier's law is no longer appropriate. An additional time derivative of the heat flux term was introduced in the formulation of the rate equation independently by Cattaneo [1] and Vernotte [2]. The combination of Cattaneo's equation with the energy equation yields HHE, which has a dependence on the second derivative of temperature with respect to time. HHE is thus a damped-wave equation whose solution results in a thermal wave propagating through the medium at a finite speed. This thermal wave is different from periodic heating in conventional heat conduction. Therefore, it may be more appropriate to describe the solution of HHE as a *nonequilibrium temperature wave*, or simply, *temperature wave*. There have been a number of publications in the literature that treat HHE as a more general form of heat conduction equation [3–5]. However, there exists no convincing experimental evidence yet to support the results of HHE, nor does there exist any sound theory that supports HHE with an acceptable thermodynamic or microscopic basis. Many recent studies improperly used the low-temperature experiment of second sound and the ultrafast laser heating experiment to justify the necessity of HHE [5]. Unfortunately, many publications that are against HHE only dealt with one or two aspects, and often contained similar misunderstandings as those supporting HHE. This has resulted in a continuously increasing number of publications that attempt to solve HHE or apply it to various material and geometric systems. Many studies on the thermal transport in biological systems have used HHE merely

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*Graduate Research Assistant, George W. Woodruff School of Mechanical Engineering.

†Professor, George W. Woodruff School of Mechanical Engineering; zhuomin.zhang@me.gatech.edu. Associate Fellow AIAA (Corresponding Author).

because the simple Fourier's law could not describe the experimental findings without questioning the validity of HHE.

The present paper aims at clarifying some common misperceptions of HHE in a systematic manner, with additional new insights into the microscopic theory of heat carriers and the paradox of infinite propagation speed. A brief description of HHE and its associated characteristics are given first. Then, the PHE and HHE are considered within statistical mechanics, classical thermodynamics, and irreversible thermodynamic frameworks. The lack of soundness of the well cited works that offer misleading experimental evidence to support HHE is also addressed.

II. Hyperbolic Heat Equation

The following rate equation for heat flux is known as Cattaneo's equation or sometimes referred to as the Cattaneo-Vernotte equation [1,2]:

$$\mathbf{q}'' + \tau_q \frac{\partial \mathbf{q}''}{\partial t} = -k\nabla T \tag{1}$$

This equation differs from the classical Fourier's law with the addition of the second term on the left hand side. At longer time scales this additional term becomes negligible and Eq. (1) reduces to Fourier's law. If the constant τ_q were negligibly small compared with the characteristic time, then this term would be removed and the equation would reduce to Fourier's law. The value of τ_q in Eq. (1) is generally taken to represent some average thermal relaxation time since there may be different relaxation times associated with various carriers in a medium. Physically the thermal relaxation time is a measure of the average time between two successive collisions of the heat carriers in a medium. It has been proposed that Cattaneo's equation is more general than Fourier's law and is valid at shorter time scales, as the heat flux is not assumed to be established instantaneously but with a time delay.

As early as 1867, Maxwell derived an equation similar to Eq. (1) based on gas dynamics. By assuming that the time rate of change of the heat flux would be negligible as the heat flux would establish itself very rapidly, Maxwell was then able to obtain Fourier's law from a microscopic viewpoint after dropping the time derivatives. Cattaneo derived Eq. (1) in 1948 using kinetic theory and extended it to demonstrate the finite propagation speed of heat in 1958. Later, others also derived Eq. (1) from the Boltzmann transport equation (BTE) [6] under the relaxation time assumption for phonon and electron transport. All the derivations made improper assumptions and do not justify the application of Eq. (1) to very short time scales, as will be elaborated in the next section.

The HHE can be arrived at by combining Eq. (1) with the energy equation for an elemental control volume:

$$\dot{q} - \nabla \cdot \mathbf{q}'' = \rho c_p \frac{\partial T}{\partial t} \tag{2}$$

Equations (1) and (2) can be combined to yield

$$\frac{\dot{q}}{k} + \frac{\tau_q}{k}\frac{\partial \dot{q}}{\partial t} + \nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_q}{\alpha}\frac{\partial^2 T}{\partial t^2} \tag{3}$$

which is the HHE with generation. Without heat generation, it can be simplified to

$$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t} + \frac{\tau_q}{\alpha}\frac{\partial^2 T}{\partial t^2} \tag{4}$$

Equation (4) is a damped-wave equation with a wave propagation speed given by

$$v_{\mathrm{tw}} = \sqrt{\alpha/\tau_q} \tag{5}$$

HHE predicts a wave, whose amplitude corresponds to the temperature in a medium, that will propagate through a medium at the speed of v_{tw} and its amplitude will decay rapidly as it propagates. This speed is finite as long as $\tau_q \neq 0$, indicating that the thermal signal cannot be felt at locations beyond the wave front. If τ_q is taken to be equal to the relaxation time of heat carriers, and the value of thermal conductivity is estimated from simple kinetic theory [7], then it can be shown that

$$v_{\mathrm{tw}} = v_g/\sqrt{3} \tag{6}$$

where v_g is either the Fermi velocity for electrons or the speed of sound for phonons.

If the temperature distribution in a medium does in fact obey HHE, then many phenomena associated with damped waves should be observed in temperature waves. One feature of temperature waves according to HHE would be a sharp wave front and wavelike temperature distribution, induced by a heat pulse with a duration smaller than or on the order of the relaxation time. The wave front could result in temperatures much higher than predicted by PHE and the amplitude of the wave front would decay as it travels through the medium. After a time period much longer than τ_q, the temperature distribution predicted by HHE would eventually settle to the one predicted by PHE.

To illustrate the concept of temperature wave, let us consider a 1D slab of finite thickness L with internal heat generation. The governing HHE becomes

$$\frac{1}{v_{\mathrm{tw}}^2}\frac{\partial^2 T}{\partial t^2} + \frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{1}{k}\left[\dot{q}(x,t) + \frac{\alpha}{v_{\mathrm{tw}}^2}\frac{\partial \dot{q}(x,t)}{\partial t}\right] \tag{7}$$

The solution of this equation has been given by Özişik and Vick [8] in terms of dimensionless variables. The dimensionless position η and time ξ are defined as

$$\eta = \frac{v_{\mathrm{tw}} x}{2\alpha} \quad \text{and} \quad \xi = \frac{v_{\mathrm{tw}}^2 t}{2\alpha} \tag{8}$$

A nondimensional source term is defined as

$$S(\eta,\xi) = \frac{4\alpha^2 \dot{q}}{q_0 v_{\mathrm{tw}}^3} \tag{9}$$

where q_0 represents the total energy generation over the entire slab for time from zero to infinity:

$$q_0 = \int_0^\infty \int_0^L \dot{q}(x,t)\,\mathrm{d}x\,\mathrm{d}t \tag{10}$$

Consider the case of short pulse laser heating of a medium. A very rapid laser pulse can be modeled as an instantaneous release of energy. Such a pulse can be represented by a piecewise dimensionless generation function that originated from a nondimensionless thickness $\Delta\eta$ and becomes zero for $\eta > \Delta\eta$.

$$\Delta\eta = \frac{v_{\mathrm{tw}}\Delta x}{2\alpha} \tag{11}$$

and

$$S(\eta,\xi) = \begin{cases} \frac{1}{\Delta\eta}\delta(\xi), & 0 \le \eta \le \Delta\eta \\ 0, & \Delta\eta < \eta \le \eta_L \end{cases} \tag{12}$$

where η_L corresponds to the boundary at $x = L$. The Dirac delta function is used to model the thermal energy generation as occurring instantaneously with all the energy released at $t = 0$. In essence, $\Delta\eta$ represents a region near the surface of the wall where the laser pulse is modeled as a generation term. Note that both boundaries are assumed to be adiabatic. Green's function method was used to arrive at the following solution for a single pulse [8]:

$$\theta(\eta,\xi) = \frac{1}{2\eta_L} + \frac{1}{\eta_L}e^{-\xi}\sum_{m=1}^{\infty}\cos(\lambda_m\eta)\frac{\sin(\lambda_m\Delta\eta)}{\lambda_m\Delta\eta} \times\left[\frac{\sin(\xi\sqrt{\lambda_m^2-1})}{\lambda_m^2-1} + \cos(\xi\sqrt{\lambda_m^2-1})\right] \tag{13}$$

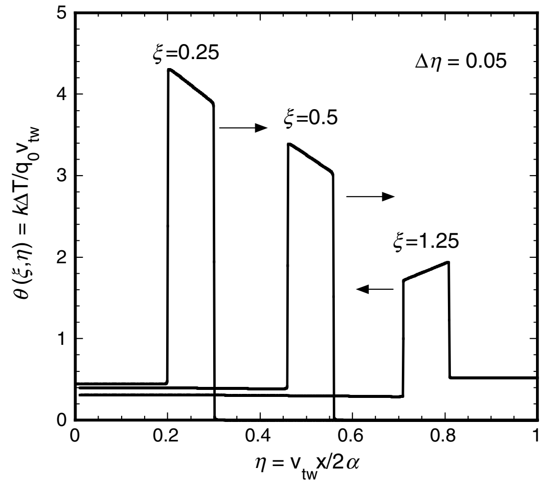


Fig. 1 Propagation of temperature wave in medium due to pulse heat source with duration less than the relaxation time.

where $\theta = k\Delta T/q_0 v_{\mathrm{tw}}$ is a dimensionless temperature with ΔT being the temperature rise with respect to the initial temperature, and the eigenvalues for the series are given by $\lambda_m = m\pi/\eta_L$.

The solution of Eq. (13) is illustrated in Fig. 1 at three dimensionless time values. The direction of wave propagation is indicated by the arrow and the decaying amplitude of the wave front. After the temperature wave front reaches the boundary, it is reflected back in the opposite direction. The wave continues to decay on each pass until the solution becomes the same as that predicted by PHE.

The possibility of resonance features of the wave in response to various source frequencies has also been investigated. Tzou [9] studied the resonance phenomena for thermal waves and predicted the critical frequency of resonance. In addition to resonance the wave may experience reflection as discussed previously, as well as refraction at the boundaries when adiabatic boundary conditions are removed. Most publications on HHE deal with numerical or analytical solutions with very few experimental studies that are questionable as will be discussed in Section V.

III. Statistical Theory

As mentioned earlier, HHE can be derived from BTE, which describes the evolution of the nonequilibrium distribution of particles in the phase space. The distribution function f describes the probability of a particle occupying a given quantum state; it is assumed that the energy levels are sufficiently dense to allow the energy distribution to be treated as if it were continuous. BTE itself has two limiting assumptions; namely, particle reactions are infrequent and the wave nature of particles can be neglected. Such conditions apply for substances such as ideal gases. Nevertheless, BTE is valid for all time and length scales and can be expressed as [10]

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}} + \mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}} = \left[\frac{\partial f}{\partial t}\right]_{\mathrm{coll}} \tag{14}$$

The last term on the right represents the effect of molecular collisions that restore the distribution function to its equilibrium state. BTE provides a method by which we can determine the behavior of nonequilibrium distributions. This is critical since heat transfer will only occur as a result of a nonequilibrium distribution. Furthermore, BTE is not limited to just gases, but can be extended to other particles and quanta that can be modeled as having infrequent interactions and obey certain distribution functions. The behavior of these particles must have some features in common with an ideal gas; thus, the term electron gas and phonon gas have been coined to describe the behavior of heat carriers in a solid medium. Cattaneo's equation can be derived from the BTE by applying the relaxation time assumption, which linearizes the collision term by assuming that the system is not too far from equilibrium so that

$$\left[\frac{\partial f}{\partial t}\right]_{\mathrm{coll}} = \frac{f - f_0}{\tau(\mathbf{v})} \tag{15}$$

Here, the term $\tau(\mathbf{v})$ is the relaxation time, which is in general dependent on the velocity or momentum of the particles. A further assumption is made that the relaxation time is independent of velocity, denoting the average relaxation time by τ_q for convenience. Another important assumption is called the local equilibrium assumption. Consider a one-dimensional problem where the temperature gradient is in the x direction only. This assumption can be expressed as follows [10]:

$$\frac{\partial f}{\partial x} \approx \frac{\partial f_0}{\partial x} \tag{16}$$

The local equilibrium assumption implies that the spatial derivative of the nonequilibrium distribution can be approximated as that of the equilibrium distribution. However, the derivation from BTE after these assumptions is not sufficient to justify HHE, because the assumption of local equilibrium does not apply to a system subject to a disturbance shorter than relaxation time when particles do not have sufficient time to interact. In fact, the HHE and the PHE are subject to the same limitation at short time scales: both are not applicable to time scales shorter or close to the relaxation time. At longer time duration when $t \gg \tau_q$, the HHE and PHE are essentially the same. Several theoretical studies have showed that neither HHE nor PHE can predict the transient behavior at short time scales for phonon or electron systems [11–13]. The statement that HHE is applicable at short time scales, as long as the characteristic length is much larger than the mean free path, is misleading. The propagation distance of the wave front during one relaxation time falls in the regime of microlength scales. Thus, local equilibrium cannot be guaranteed and HHE is not appropriate for use at time scales shorter or close to the relaxation time, which is the regime that the HHE was intended to be applied to improve the PHE.

Another way to see why local equilibrium condition is not satisfied is by observing the solutions of HHE as shown in Fig. 1. The solution has a very sharp wave front with an elevated temperature. Particles ahead of this wave front will still have an undisturbed equilibrium distribution while particles at the wave front will be in a state of nonequilibrium with an elevated temperature. The wave front will pass through this region and the particles occupying the region will then experience a sudden jump in temperature. Obviously, there is no way to establish local equilibrium at the front and back end of the pulse. Körner and Bergmann [14] showed that for 3-D problems involving a point heat source, HHE can result in nonphysical solutions, such as a local negative energy value and temperature below absolute zero. Many also showed that HHE could predict results that contradict the second law of thermodynamics as will be discussed in the next section.

An apparent advantage of HHE is the removal of the paradox of infinite speed of propagation associated with Fourier's law. It is interesting to note that the classical and quantum statistics of particles do not explicitly limit the speed of particles. The probability of gas particles and free electrons in their statistical models allows large velocities with a probability approaching zero as the velocity goes to infinity. The statistical model of phonons, however, will have a finite velocity limit that depends on the speed of sound in the material but allows an unlimited energy per particle although the probability of such high energies quickly approaches zero as the phonon energy increases. The free path distribution allows for a particle to travel an extremely large distance without colliding with another particle with a very small probability. Take for instance the Maxwell-Boltzmann distribution; the probability is nonzero for a particle to travel faster than the speed of light, albeit the probability is extremely small. Any particle traveling at a finite speed, even if that speed is greater than the speed of light, has a small probability of having a relaxation time long enough to allow a miniscule but nonzero effect at very large distances. Under the well-accepted statistical models, heat could propagate at an unbounded rate, although the probability of carriers propagating at an extremely high rate decreases to nearly zero. The limitation is set by the negligible probability; hence, these statistics will not predict any meaningful heat propagation at a speed greater than the speed of light that will contradict the special theory of relativity. A finite but negligibly small change is exactly what should be expected by any equation that is consistent with the statistical models of particles. This is essentially the prediction of Fourier's law which states that the instantaneous response to a thermal disturbance at an infinite distance is not zero but infinitesimally small. It is PHE, rather than HHE, that gives results that are fundamentally consistent with statistical mechanics.

Note that Fourier's law is only an approximate model of heat transfer that describes the diffusion processes in a single-phase material not involving phase change, advection, or bulk flow. Moreover, all measurements of temperature and heat flux are subject to instrumental uncertainties, as well as random noises existing everywhere in nature. The statement that Fourier's law predicts an infinite speed of propagation is strictly limited to the sense of an infinitely small temperature change. The thermal response predicted by Fourier's law decays rapidly toward large distances to below any meaningful values. A more reasonable definition of the speed of heat propagation is the diffusion speed, which is generally much lower than the thermal wave speed given in Eq. (5) as to be elaborated in Sec. V.

Earlier pioneers like Cattaneo, Vernotte, and Taverneir did not examine the validity of HHE. When the thermal wave phenomena was actually observed at lower temperatures, many believed that Cattaneo's equation should be a generalized Fourier's law [3]. It is important to differentiate between HHE and similar equations that have physical foundations such as the two-temperature model (TTM) for ultrafast laser heating and the thermal wave phenomena at low temperatures. TTM, unlike HHE, may depict certain physical phenomena that arise from coupling due to the interactions of electrons and phonons. The assumption is made that phonons are in equilibrium with phonons and that electrons are in equilibrium electrons, but the two types of particles are not in equilibrium with each other [10]. Thus, TTM gives two parabolic differential equations that are coupled with each other, and both equations are consistent with Fourier's law. Results corresponding to the prediction of TTM have been obtained experimentally for femtosecond pulsed-laser interaction with metals [15]. Furthermore, TTM does not predict any wavelike features nor does it predict a finite speed of propagation of a thermal signal.

Low-temperature behavior of thermal waves in liquid helium and a few dielectrics is fundamentally different from HHE, although both predict the second sound (or thermal wave velocity) as described in Eq. (5). This behavior is actually due to two different scattering mechanisms for phonons, one associated with a normal process in which momentum is conserved and an Umklapp process in which the momentum is not conserved. Guyer and Krumhansl [16] derived the dispersion relation for second sound in solids based on BTE, where the collision contribution is approximated by two terms. The first term is given as

$$-\frac{f - f_\lambda}{\tau_N} \tag{17}$$

which is associated with the normal scattering process. Since phonon momentum is conserved, the normal process will not change the direction of energy flow but it can change the distribution function. The change of the distribution function due to the normal process will be zero when the distribution function is the same as that of a uniformly drifting phonon gas f_λ. The second collision term represents the Umklapp process and is given as

$$-\frac{f - f_0}{\tau_0} \tag{18}$$

The Umklapp scattering process does not conserve momentum and will eventually return the distribution function to the equilibrium distribution function f_0. The two relaxation times are the relaxation times associated with each of the scattering processes. Combining the scattering effects gives a simplified BTE for phonon scattering

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = -\frac{f - f_\lambda}{\tau_N} - \frac{f - f_0}{\tau_0} \tag{19}$$

The solution of this linearized BTE gives the following differential equation for the effective temperature of phonons [3,16]:

$$\nabla^2 T + \frac{9\tau_N}{5}\frac{\partial}{\partial t}\nabla^2 T = \frac{3}{\tau_0 v_a^2}\frac{\partial T}{\partial t} + \frac{3}{v_a^2}\frac{\partial^2 T}{\partial t^2} \tag{20}$$

where v_a is the average phonon speed. When the condition $\tau_N \ll \tau_0$ is satisfied, the energy transfer is dominated by wave propagation. At higher temperatures, the scattering rate for the U processes is usually very high, and the N processes contribute little to the heat conduction or thermal resistance. Therefore, the reason why temperature waves have never been observed in dielectric crystals above 100 K is not because of their small relaxation time, in the range from 10^{-10} to 10^{-13} s, but because of the lack of mechanisms required for a second sound to occur. No experiments have ever shown a second sound in metals, as suggested by HHE. Note that Eq. (20) predicts a wavelike feature with a diffusion tail that is subject to the same criticism of infinite speed or propagation. A very illustrative summary of solutions to the various heat conduction models was given by Tang and Araki [17]. Only the pure HHE has a finite speed of propagation, however, it is contradictory to statistical mechanics as discussed earlier and violates the laws of thermodynamics as to be discussed next.

IV. Thermodynamic Consideration

The zeroth law of thermodynamics states that two bodies that are in thermal equilibrium with a third body are in thermal equilibrium with each other. This statement implies the existence of a property of the system known as temperature. This definition of temperature requires that the particles in a body are in a state of interaction with one another. A single particle in the body possesses energy but the temperature of a single particle cannot be defined. Within any given medium, particles will distribute themselves at different energy levels depending on the temperature. Temperature in this sense only possesses meaning on scales large enough to define an average energy of particles and only when there is sufficient interaction between particles to define a macroscopic temperature. Therefore, local equilibrium is essential for the definition of thermodynamic temperature to apply [18].

In this sense, the meaning of temperature described in HHE does not satisfy the definition of temperature in thermodynamics. Note that τ_q is a measure of interaction in the medium and presumably to be the same or at least on the same order of the relaxation time in the medium. At time scales near or less than τ_q, there is simply insufficient time for particle interactions to establish themselves. A nonequilibrium temperature can be defined based on the energy of the particles, but this is different from the definition of temperature and such a temperature cannot be measured by conventional means such as a thermocouple or electrical resistance thermistor. Many researchers did not make this distinction between the thermodynamic temperature and nonequilibrium temperature when studying HHE and the associated phenomena.

If the temperature in HHE were taken in the classical sense there could be a negative entropy generation according to classical thermodynamics, which is in violation of the second law. In some cases, HHE may predict a heat transfer from the cold region to the hot region [19]. Barletta and Zanchini [20] derived an expression for the rate of entropy generation based on Eq. (1) using classical thermodynamics definitions:

$$\dot{S}_{\mathrm{irr}} = \frac{1}{kT^2}\mathbf{q}'' \cdot \left(\mathbf{q}'' + \tau_q \cdot \frac{\partial \mathbf{q}''}{\partial t}\right) \tag{21}$$

When the heat flux decreases rapidly enough with respect to time, Eq. (21) will take on a negative value, which is forbidden in classical thermodynamics as this would indicate a decrease in entropy due to heat dissipation.

Hence, the temperature in HHE must be interpreted as a nonequilibrium temperature. Several nonequilibrium thermodynamics theories have been proposed which would give the entropy generation a positive value when applying HHE. The basic argument is that if HHE is a correct physical model, one must modify the definition of entropy or internal energy of classical thermodynamics to justify HHE within a modified thermodynamics [21]. These so-called nonequilibrium thermodynamics propose that entropy is dependent on dissipative fluxes such as the heat flux, in addition to the classical thermodynamic variables. The most widely known nonequilibrium thermodynamics that justifies HHE is called extended irreversible thermodynamics (EIT) by Jou et al. [22]. It should be noted that the original irreversible thermodynamics proposed by Onsager [23] dealt with coupled transport phenomena such as thermoelectricity and is an extension of Fourier's law. Onsager explained that his theory is not meant to be applied at time scales smaller than the relaxation time. The purpose of EIT is to extend macroscopic thermodynamics to nonlocal equilibrium situations. EIT introduces dissipative fluxes as independent variables in addition to the classical thermodynamic variables to describe a system on time scales when these terms have not decayed, but that a sufficient amount of time has elapsed to allow the system to be described by only the dissipative fluxes and the classical thermodynamic variables. Jou and coworkers derived an equation for a generalized Gibbs equation and an expression for the entropy generation [22]. They further assumed that the nonlocal-equilibrium entropy generation must be positive. This assumption is used to limit the values that the dissipative fluxes can assume to meet the requirements of the second law under EIT. The mathematical details are not covered here, but a thorough overview of EIT can be found in the literature [22]. The only justification for EIT as a valid description of any real processes seems to be HHE, while at the same time the only theoretical justification for HHE seems to be EIT. Hence, EIT cannot be justified until HHE has been verified experimentally. A few papers contain experimental evidence of HHE and have been well cited by others as validation of HHE. Although these experiments have been questioned by many others, a critical examination is necessary to complete this article.

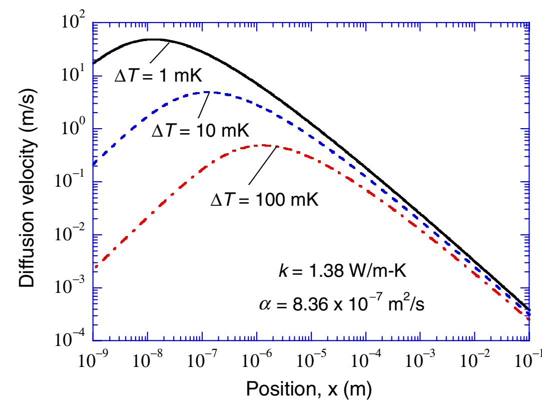
V. Experimental Observations

Kaminski [24] performed experiments with several nonhomogenous materials (one of which was sand). Placed in the sand were an electric heating wire surrounded by electric insulation inside of a needle and a thermocouple inside of another needle used to measure the temperature. The experimental setup involved a thermocouple placed 6.8 mm and 16.8 mm from a restive heating source and found that heat propagated in sand at a velocity of $0.143\ \mathrm{mm/s}$ and τ_q for sand to be approximately 20 s. The value for relaxation time and velocity were determined when the penetration depth of the heat source reached the thermocouple. The penetration depth of a heat source is defined as the distance inside of the medium at which the temperature change has reached a certain threshold. This value of τ_q seems artificially high with regard to HHE and has no physical meaning as it is too large to represent an average time between scattering of particles. It is also possible that the solution that Kaminski obtained from Fourier's law was incorrect due to improper formulation of the problem or some other phenomena associated with a porous media such as sand grains. It should be noted that, although Fourier's law predicts the effect of a heat flux to be instantaneously felt everywhere throughout the medium, an effective propagation speed of heat can be defined based on the penetration depth. Even under Fourier's law this effective propagation speed can be very slow as the effect at large distances will be negligible (below the sensitivity of the measurement instrument) until a sufficient amount of time has passed. Even though in theory the temperature will change a finite amount at large distances, the change will not be detectable by experimental means. Hence, the velocity that Kaminski observed may simply be an effective velocity of heat predicted by Fourier's law.

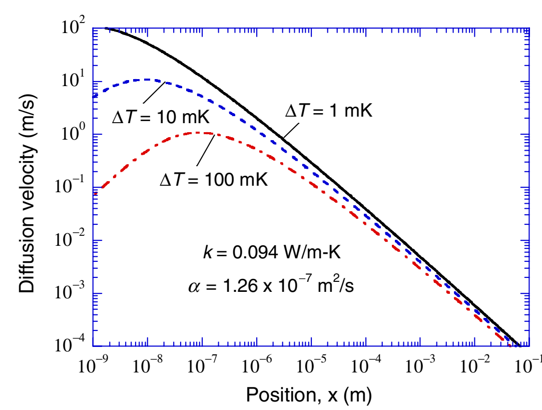
To demonstrate just how slow a diffusion process is, let us take for instance a semi-infinite slab with constant surface heat flux q_s''. This problem has been solved in Carslaw and Jaeger [25], and the temperature distribution is given as

$$T(x,t) - T_i = \Delta T = 2q_s'' \frac{\sqrt{\alpha t}}{k}\left(\frac{\exp(-\zeta^2)}{\sqrt{\pi}} - \zeta \operatorname{erfc}(\zeta)\right) \qquad (22)$$

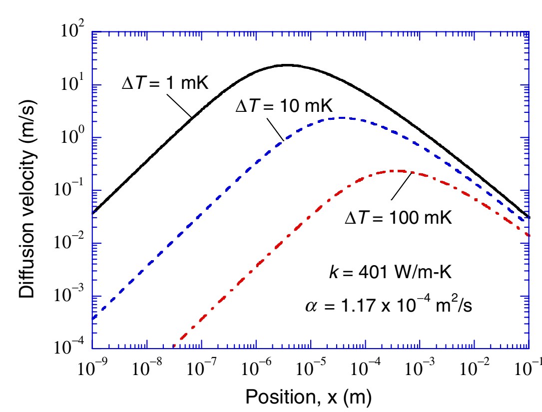
where $\zeta = x/\sqrt{4\alpha t}$ is a similarity parameter and T_i is the initial temperature of the medium. The effective penetration depth can be defined based on a minimum temperature change in the medium. The diffusion velocity can be found by dividing the effective penetration depth by the elapsed time. The effective diffusion speed is shown in Fig. 2 for a semi-infinite medium with the thermal properties of a) fused silica; b) hardboard siding; and c) copper, respectively, at



a) Fused silica



b) Hardboard siding



c) Copper

Fig. 2 Velocity of diffusion for a semi-infinite medium with a constant surface flux of $q_s'' = 2 \times 10^5\ \mathrm{W/m^2}$: (a) fused silica ($SiO_2$); (b) hardboard siding; (c) copper. The thermophysical properties of these materials are taken from [26].

300 K [26]. Three curves show three different temperature change thresholds used to define the effective penetration depth. At a distance of 20 mm inside of the medium the effective thermal velocity for a temperature rise of $\Delta T = 1$ mK will result in an effective thermal velocity of about 1.67 mm/s for fused silica, 0.31 mm/s for hardboard siding, and 122 mm/s for copper. The effective velocity of a diffusion process may vary with the geometry, position, heat flux, minimum temperature rise, and properties of a material. Nevertheless, it can be seen that the diffusion process results in very slow thermal signal propagation, contrary to the belief of many researchers. As early as in 1997, Tzou [27] pointed out that the thermal wave model is unsuitable for describing transient thermal behavior in casting sand (or other nonhomogeneous materials). However, many recent publications on heat transfer in biological systems continue to misinterpret the speed of diffusion as the speed of temperature waves in HHE.

Mitra et al. [28] performed an experiment on processed meat to show that temperature waves have a finite propagation speed in certain biological media and other materials with nonhomogenous inner structures. Two samples held at different initial temperatures of identical processed meat were brought into contact and the temperature was measured by a thermocouple embedded in each of the samples. The experimental results showed a temperature jump, after a finite amount of time had passed, rather than a smooth transition as would be predicted by Fourier's law. In addition, two other experiments were performed to demonstrate phenomena associated with temperature waves such as superposition of two temperature waves and one additional experiment intending to demonstrate the finite propagation speed of temperature waves in a medium. In one of their experiments, a slab (14.3°C) was sandwiched between a large cold sample (6.2°C) and a large hot sample (24.1°C). Results shown in the paper indicated a superposition phenomena of the temperature waves as measured by a thermocouple embedded in the thin meat sample. If the temperature of HHE is an effective nonequilibrium temperature, it seems unlikely one would be able to measure this temperature with a simple thermocouple. Their results were never repeated by themselves or verified by others.

A similar experiment was performed by Herwig and Beckert [29] who used water flowing through a copper pipe in a box filled with a desired medium. The experiment was conducted with both sand and processed meat similar to the experiments used to validate HHE [24,28]. The results obtained in [29] showed that the temperature distribution predicted by Fourier's law was within the uncertainty of the temperature values obtained in their experiment.

If the experimental results validating HHE were reliable, then there would be a plethora of similar findings in the literature. Such results, however, are lacking. It seems unlikely that nonporous media would sometimes obey HHE and sometimes obey PHE at such relatively long time scales as in the aforementioned experiments. On a microscopic basis, diffusionlike behavior would be expected and any disagreement may be due to other phenomena associated with the nonhomogeneous inner structure not related to HHE. At the present time, conclusive experimental results supporting HHE do not exist.

Finally, it should be reminded that the experimental results showing thermal waves in liquid helium and cryogenic dielectric crystals are not a result of HHE, but are rather due to another phenomena. This phenomena is only observable at low temperatures where the normal process of phonon scattering becomes more significant. This gives rise to two relaxation mechanisms for phonon scattering, one associated with the normal process and one with the Umklapp process, leading to a two-relaxation time model. This model is only valid when the time scale is greater than the relaxation time of the normal process and less than the combined relaxation time of both processes [10].

VI. Conclusions

The hyperbolic heat equation is fundamentally inconsistent with statistical mechanics and the solution violates the laws of thermodynamics. The claim that HHE can be derived from BTE is misleading since the assumptions made during the derivation invalidates its application to time scales shorter than the relaxation time, for which HHE was intended. While Fourier's law has its own limitations, HHE does not extend Fourier's law to describe the heat transfer process at short time scales, as it was originally proposed and believed by many heat transfer researchers. The infinite speed of propagation of diffusion process is consistent with the statistical theory and, furthermore, Fourier's law predicts a very slow diffusion process that has been validated by numerous experimental observations. Thermal waves observed at low temperatures are associated with a different phenomena of two-relaxation times and does not serve as evidence of HHE. The two-temperature model for ultrafast laser heating of metals contains a pair of coupled PHEs and does not predict wavelike behavior. The few experiments that showed temperature waves at near room temperatures in nonhomogenous materials could not be reproduced by others. Furthermore, attempts to thermodynamically justify HHE have not themselves been justified, albeit the completeness and self-consistency of these theories. It is hoped that this study will clarify some common misunderstandings about HHE in the heat transfer community to avoid similar pitfalls in the future.

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